

Chaotization inside Quantum Black Holes

Andrea Addazi^{1,2}

¹ *Dipartimento di Fisica, Università di L'Aquila, 67010 Coppito AQ, Italy*

² *Laboratori Nazionali del Gran Sasso (INFN), 67010 Assergi AQ, Italy*

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We show how the horizon geometry and entropy of a Semiclassical Black Hole can be reconstructed from a system of $N \gg 1$ horizonless conic singularities with average opening angle at the horizon $\langle \Theta \rangle = 2\pi$. This conclusion is strongly motivated by a generalized Wheeler-De Witt equation for quantum black holes. We will argue how infalling information will be inevitably chaotized in these systems. A part of the initial probability density will be trapped inside the system, in back and forth scatterings among conic singularities, for a characteristic time close to the Semiclassical BH life-time. Further implications on information paradoxes are discussed.

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1. INTRODUCTION AND CONCLUSIONS

The information paradox of semiclassical black holes ¹ could suggest us that "Nature abhors real horizons" ^{2 3}. However, naked singularities seem to be unstable solutions with respect to external electromagnetic, gravitational and matter perturbations [8]. So that, Nature seems also to abhor naked singularities! On the other hand, numerical simulations of stellar collapses seem to inevitably lead to naked singularities [9].

Recently, we have suggested that information is chaotized inside realistic black holes, thought as a system of horizonless geometries [10, 11]. This approach could be a step toward the solution of the puzzle mentioned above. In this paper, inspired by these our previous ones, we suggest that the BH quantum state as a superposition of the wave functions of *a large number of conic naked singularities*. We will argue how a semiclassical BH solutions are asymptotic limits of a $N \rightarrow \infty$ of conic singularities (randomly oriented). We will show how the geometry of a BH is effectively recovered by this horizonless system of conic singularities! ⁴ In other words, the thermodynamical properties of semiclassical black holes are recovered by an ensemble of conic singularities except for small correction to Bekenstein-Hawking entropy. This conclusion is formally motivated by a generalized Wheeler-De Witt equation for quantum black holes [12–14] based on the concept of Wald entropy [15]. An alternative argument based on euclidean path integral was given in [10, 11] and it can also be extended to our new *ansatz* considering conic singularities (see appendix B).

Let us suppose a thought scattering of a plane wave function on a system of N conic singularities ⁵. Such a plane wave will be scattered among the conic geometries. The non-relativistic quantum mechanical scattering problem of an incident wave function on a conic space-time can be analytically solved. In particular, we will see how the scattering amplitude can be expressed as a simple combination of Bessel and Henkel functions. However, sequential scatterings of a wave function on a large number of randomly oriented conic singularities will lead to a chaotization of information. In fact the resultant wave function is a superposition of the incident wave function with ones diffracted by each "scatterators". This can be thought as a wave function scattering on a *quantum Sinai billiard*! ^{6 7}

Infalling information is highly chaotized inside the space-temporal billiard. What one will expect is that the initial probability will be fractioned into two contributions. In fact, a part of the initial probability density will "escape"

¹ See [1–6] for classical references on these subjects.

² This is a citation from the title of paper [7].

³ Some extensions of general relativity are plagued by inconsistencies at classical level. For example in [22], we have discovered geodesic instabilities in a branch of black holes' solutions previously suggested in massive gravity.

⁴ This could also have implications in astrophysics. As discussed in [37–39], naked singularities could be detected by virtue of their very peculiar signatures in gravitational lensing measures. On the other hand, our systems of conic naked singularities could be differentiated by semiclassical black holes if their "frizzyness is large" enough to be measured in gravitational lensings.

⁵ Because of this paper is a part of a special dedicated to Einstein and Bohr, we retain appropriate to celebrate these genial theoretical physicists with a "gedanken experiment" -that we hope it can provide a "breakthrough" conclusion about information fate inside a black hole, or at least it can stimulate interesting counter-arguments against our one.

⁶ See [17–21] for useful references in quantum chaos theory.

⁷ Different applications of chaos theory in black holes' physics were suggested in [16].

by the system while a part will remain "trapped" forever in the system because of back and forth scatterings, *i.e* for all the system life-time. The formation of trapped chaotic saddles inside billiards seems inevitable. In classical chaotic systems, these correspond to surfaces of unstable orbits, while in quantum system they correspond to a chaotic superposition of unstable wave functions. This problem is treated with a quantum semiclassical approach in our previous contributions [10, 11], reviewed in Appendix C. In this paper we will show formalities of the same problem in Born approximation.

So, infalling quantum pure states are fractioned into a "forever" trapped state $|TRAPPED\rangle$ and an emitted one $|B.H.\rangle$ (in form of Bekestein-Hawking radiation):

$$|IN\rangle = c_1|B.H.\rangle + c_2|TRAPPED\rangle$$

where $c_{1,2}$ are complex coefficients depending on the particular configuration of conic singularities, with

$$|c_1|^2 = |\langle B.H.|IN\rangle|^2$$

$$|c_2|^2 = |\langle TRAPPED|IN\rangle|^2$$

$$|c_1|^2 + |c_2|^2 = 1$$

Principles of quantum mechanics not allow a transition with $|c_1|^2 = 1$, because of $|B.H.\rangle$ is in a mixed entangled state, while $|IN\rangle$ is supposed in a pure one. However, a combined state of $|B.H.\rangle$ and $|TRAPPED\rangle$ can be a pure one. In this case, a transition from a $|IN\rangle$ state to a pure combination of two mixed states $|B.H.\rangle$ and $|TRAPPED\rangle$ is allowed by unitary evolutions. During the black hole life-time, $|TRAPPED\rangle$ is not accessible to an ideal external observer, so that to reconstruct the initial pure state from this one is practically impossible. So that, a quantum mechanical approach describing the unitary evolution of wave functions in time has not sense, in this system. A wave functions' approach can be substitute by a quantum statistical mechanics' approach in terms of density matrices.

However, let us remark that quantum field theory corrections to the non-relativistic approach will ulteriorly favor the chaotization of infalling information. Quantum fields' interactions are crucially important in our system. In fact they will "mediate" a new form of quantum dechoerence induced by the non-trivial configuration of the space-time. Let us consider the (famous) thought experiment of a Bekenstein-Hawking pair created near the horizon, one infalling and one tunnelling out. Of course, they are entangled and this will lead to the (famous) firewall paradox in a semiclassical black hole. What happen in our space-temporal Sinai billiard? The infalling particle will be chaotized back and forth among asperities and it will start a complicated cascade inside. In fact, in non-trivial background (thought as a superposition of gravitons) $\langle G...G \rangle$, infalling particles can inelastically scatter on it: for example an inelastic scattering of an electron can create electromagnetic or hadronic cascade as

$$e^- + \langle G.....G \rangle \rightarrow e^- e^+ e^- + \langle G.....G \rangle; \quad e^- + \langle G.....G \rangle \rightarrow e^- q\bar{q} + \langle G.....G \rangle$$

$$e^- + \langle G.....G \rangle \rightarrow e^- \gamma + \langle G.....G \rangle; \quad e^- + \langle G.....G \rangle \rightarrow e^- g + \langle G.....G \rangle; \quad$$

and so on, depending on the particular background structure and local CM energy of collision ⁸. Iterating chaotic back and forth scattering and fields' interactions, the initial external particle will be no more entangled with one one partner but with a very large and chaotic system of particles. However, this practically means that such a particle is disentangled. One can also estimate the entanglement entropy in this system. In particular, considering a system of P Bekenstein-Hawking couples cascading inside the system, they will generate $N \gg P$ particles, exponential increasing with the number of collisions inside the system. Let us suppose for simplicity that a fixed number N of particles are produced after n processes, in turn producing a rate of $\langle \bar{\nu} \rangle P$ particles for each process. In this case one can estimate the entanglement entropy inside the system as

$$S_{e.e} = -\text{Tr} \rho_{\text{INSIDE}} \log S_{\text{INSIDE}} \sim n \log P$$

⁸ For the moment, we only consider standard model interactions. However, in presence of non-perturbative interactions induced by exotic instantons, particles' cascades could also violate B/L numbers [25–34] On the other hand, new non-local interactions can emerge in the cascade near the effective non-local scale. See [23, 24] for discussions on these aspects.

Another further question regards the fate of such a system, considering its (semi) Bekenstein-Hawking evaporation. The emission of trapped probability density $\rho(T)$ is approximately described by

$$\frac{d\rho(T)}{dT} \sim -\frac{1}{T^2} e^{-\Gamma(T)T}$$

where $\rho(T)$ is the trapped probability density, dependent by the number of asperities N_s as $\rho \sim N_s e^{-\Gamma(T)T}$, where Γ parametrizes the *effective average deepness* of asperities (trapping ρ); and the number of asperities N_s is in turn dependent by the Black hole mass evolution $dM/dT = -1/8\pi T^2$. As a consequence, the trapped information will be exponentially re-emitted in the environment for $\Gamma(T) \rightarrow 0$. As a consequence, an S-matrix describing the entire black holes' life $\langle \text{collapse} | S | \text{complete evaporation} \rangle$ is unitary.

We conclude that these new interpretation of quantum black holes as a large ensemble of conic naked singularities seem a viable way-out from information paradoxes, leading to intriguing chaotic billiard-like dynamical effects in its interior. However, this approach remains incomplete. For example, we cannot compute Bekenstein-Hawking entropy from this approach and the physical interpretation of conic naked singularities remain unknown. In other words, an UV completion of our model is still unexplored⁹.

2. QUANTUM BLACK HOLES AND WAVE FUNCTIONS

A possible reinterpretation of Quantum Black Holes is based on the following quantum mechanical postulate: the quantum state of a black hole is described by a quantum wave function $|\Psi\rangle$. This implies that the BH entropy is described by a wave function $\langle S_w | \Psi \rangle = \Psi(S_w)$. In Appendix A, we discuss mathematical formalities of this approach, with its references¹⁰.

BH wave functions on entropy representation is described by

$$\frac{1}{i} \frac{\partial \Psi(S_w)}{\partial S_w} = \Theta \Psi(S_w) \quad (1)$$

Now, we can perform a semiclassical approximation. We demand that our wave function will be centered into the average value $\langle S_w \rangle = \frac{1}{4} A_H$. The equation will take the typical semiclassical form

$$\frac{1}{i} \frac{\partial \Psi(S_w)}{\partial S_w} = \Theta_{WKB} \Psi(S_w) \quad (2)$$

where

$$\Theta_{WKB} = \left[2\pi - \frac{1}{iC_1} (S_w - \langle S_w \rangle) + \dots \right] \quad (3)$$

Clearly, such an approximation can be accepted if the BH has a large entropy. Semiclassical equation has, as usual, a gaussian solution

$$\Psi(S_w) = N_1 e^{-2\pi i S_w} e^{-\frac{1}{2C_1} (S_w - \langle S_w \rangle)^2} \quad (4)$$

where N_1 is an opportune normalization. Sol.(4) has a clear interpretation: in the semiclassical limit, fluctuations around the "saddle point" are Gaussian distributions. So that C_1 is just

$$C_1 = 2\Delta S_w^2 = 2\langle (S_w - \langle S_w \rangle)^2 \rangle \quad (5)$$

so that Sol.(6) can be rewritten as

$$\Psi(S_w) = N_1 e^{-2\pi i S_w} e^{-\frac{1}{4\Delta S_w^2} (S_w - \langle S_w \rangle)^2} \quad (6)$$

⁹ It is possible that these conic geometries are sustained by topological defects or exotic non-perturbative configurations. For example, supercritical cosmic strings generate conic naked singularities [40, 41].

¹⁰ The formalism that we will use was suggested in several papers (see Appendix A). Here, we consider a different interpretation of generalized Wheeler-De Witt equations.

In the dual Fourier space, one can find out a gaussian distribution also for Θ :

$$\tilde{\Psi}(\Theta) = N_2 e^{i\langle S_w \rangle \Theta} e^{-\frac{1}{2\sigma_2}(\Theta - \langle \Theta \rangle)^2} \quad (7)$$

where $\langle \Theta \rangle = 2\pi$. Θ and S_w are conjugated variables satisfying the indetermination principle $\Delta\Theta\Delta S = \hbar/2$. (4) is interpreted as wave function for a semiclassical black hole.

However, the same result can be re-obtained as a superposition of a large number number of quantum wave functions ψ_Θ with fixed values of Θ . Among the infinite samples reproducing (6), a lot of possible wave functions ψ_Θ will have $\Theta \neq 2\pi$. But from the geometric point of view, a $\psi_{\Theta \neq 2\pi}$ describes an horizonless conic singularity. Such a conic singularity along the z-axis has a metric

$$ds^2 = -dt^2 + dr^2 + \left(1 - \frac{\Psi}{2\pi}\right)^2 r^2 d\psi^2 + dz^2 \quad (8)$$

where Ψ is the deficit angle, related to the opening angle as $\Theta = 2\pi - \Psi$.

Let us suppose a sample of N conic singularities described by the N entropy variables $S_w^{(1)}, S_w^{(2)}, \dots, S_w^{(N)}$, with corresponding wave functions $\psi_{\Theta_1}(S_w^{(1)}), \psi_{\Theta_2}(S_w^{(2)}), \dots, \psi_{\Theta_N}(S_w^{(N)})$. For $N \gg 1$, the central limit theorem will guarantee that a Random Variable

$$S_w = \sum_{i=1}^N S_w^i$$

will be distributed as gaussians. In order to recover a wave function (6), we have to impose only one condition:

$$\langle S_W \rangle = \frac{1}{N} \sum_{i=1}^N S_W^i \quad (9)$$

corresponding to

$$\langle \Theta \rangle = \frac{1}{N} \sum_{i=1}^N \Theta_i$$

in the dual space ¹¹.

Clearly, relation (9) are expected to be only an approximated one. So that, in principle one could distinguish a semiclassical black hole to a "fictious" one by small deviations by Bekenstein-Hawking entropy, *i.e* by deviations from the central value of (6):

$$|\Delta S| = \left| \frac{1}{N} \sum_{i=1}^N S_W^i - \langle S_W \rangle \right| \ll \langle S_W \rangle \quad (10)$$

As a consequence, a system with a large number of horizonless conic singularities can have the same entropy of a Quantum Black Hole, in semiclassical limit, with small corrections from thermality. Their wave functions are not entangled, *i.e* their associated metrics have to be non-interacting. This approximation is reasonable in semiclassical regime, in which gravitational interactions among metrics are strongly suppressed as well as exchanges of matter entropy among metrics.

In the following sections, we will study what happen to a pure state falling toward a system of N conic singularities.

¹¹ Of course, examples of distributions ψ_{Θ_i} avoiding central limit theorem, like Chauchy-Lorentz one cannot be considered, or at least they can only be a small fraction of distributions in the large ensemble of naked singularities. In particular, we remind that for C.L. distributions, moments are undefined. So that, all examples of distributions with these kind of pathologies cannot contribute to the formation of a black hole.

3. NON-RELATIVISTIC QUANTUM SCATTERING

3.1. Scattering on a single cone

Let us consider the Schroedinger equation for a particle, in a cone geometry ¹².

$$i\frac{\partial}{\partial t}\psi(x) = -\frac{\Delta_c}{2m} + A\frac{\delta(r-\bar{r})}{r} \quad (11)$$

where Δ_c is the Laplacian in the conical geometry. For simplicity, we have considered a cone with its axis coincident with the z-axis. In fact, the radius of the cone boundary is $r = \bar{r}$, and it can be encoded in the equation as a δ -potential, while A is the dimensional "coupling" of the potential.

As usually done for this type of problem, we can separate the variables as

$$\psi(t, x) \sim e^{-i\omega t} \phi_n(r) (\sin n\nu\theta, \cos n\nu\theta)^T, \quad n = 0, 1, 2, \dots \quad (12)$$

and defining the adimensional parameter $a = 2mA$ and substituting (12) to (11) we obtain

$$\frac{d^2\phi_n(r)}{dr^2} + \frac{1}{r}\frac{d\phi_n(r)}{dr} + \left[k_z^2 - \frac{n^2\nu^2}{r^2} - \frac{a}{r}\delta(r-\bar{r})\right]\phi_n(r) = 0 \quad (13)$$

We demand as contour conditions

$$\phi_n(a + o^+) - \phi_n(a + o^-) = 0 \quad (14)$$

so that we can map such a problem to another free-like equation

$$\frac{d^2\phi_n(r)}{dr^2} + \frac{1}{r}\frac{d\phi_n(r)}{dr} + \left(k_z^2 - \frac{n^2\nu^2}{r^2}\right)\phi_n(r) = 0 \quad (15)$$

This equation can be also rewritten as

$$\frac{d^2u_n(r)}{dr^2} + \left(k_z^2 - \frac{n^2\nu^2}{r^2}\right)u_n(r) = 0 \quad (16)$$

where $u_n = r\phi_n$ and k_z^2 .

The solution (regular) corresponding to the continuous part of the spectrum is

$$\phi_n(r) = c_n^0 J_{n\nu}(k_z r), \quad r < \bar{r} \quad (17)$$

$$\phi_n(r) = c_n^-(k_z)H_{n\nu}^-(k_z r) - c_n^+(k_z)H_{n\nu}^+(k_z r), \quad r > \bar{r} \quad (18)$$

These solutions are valid for all values of a in the δ -potential. Our problem has two matching conditions

$$c_n^0(k_z)J_{n\nu}(k_z\bar{r}) = c_n^-(k_z)H_{n\nu}^-(k_z\bar{r}) - c_n^+(k_z)H_{n\nu}^+(k_z\bar{r}) \quad (19)$$

$$c_n^0(k_z) \left[\frac{a}{k_z\bar{r}} J_{n\nu}(k_z\bar{r}) + J'_{n\nu}(k_z\bar{r}) \right] = c_n^-(k_z)H'_{n\nu}^-(k_z\bar{r}) - c_n^+(k_z)H'_{n\nu}^+(k_z\bar{r}) \quad (20)$$

(prime is the differentiation with respect to the adimensional variable $k_z r$).

This problem can be viewed as a scattering one. The corresponding solution for the S-matrix is

$$S_n(k_z) = \frac{aJ_{n\nu}(k_z\bar{r})H_{n\nu}^-(k_z\bar{r}) + 2i/\pi}{aJ_{n\nu}(k_z\bar{r})H_{n\nu}^+(k_z\bar{r}) - 2i/\pi} \quad (21)$$

¹² Perhaps this problem could be found in standard test of advanced quantum mechanics and non-relativistic quantum scattering theory. I have not found any useful references about this particular problem of quantum scattering, so that I have just decided to repeat the exercise in all the details.

related to f_n as usual:

$$S_n = 1 + 2ik_z f_n$$

so that

$$|S_n| = 1 \rightarrow S_n = e^{2i\delta_n}$$

We also remind as f_n is related to this phase δ_n :

$$f_n = \frac{e^{2i\delta_n} - 1}{2ik_z} = \frac{e^{i\delta_n} \sin \delta_n}{k_z}$$

Let us remind that, as usual, the asymptotic expansion of the radial part of the wave function can be written as the sum of the incident plane-wave on the conic geometry and the spherical one as

$$\frac{1}{(2\pi)^{3/2}} \left[e^{ik_z z} + f(\theta, \phi) \frac{e^{ikr}}{r} \right]$$

3.2. Non-Relativistic Quantum Scattering on a Space-time Sinai Biliard

Let us consider a series of scatterings on a large number of N cones, disposed with a uniform random distribution of axis. Let us suppose a box of $n \times m \times p$ cones, n in the x-axis, m in y-axis, p in z-axis (not necessary disposed as a regular lattice). Let us call $\mathcal{N}_1, \mathcal{N}_2$ the sides sited in the xy-planes, $\mathcal{M}_{1,2}$ in xz-planes, $\mathcal{P}_{1,2}$ in zy-planes, edges of the box of cones. Suppose an incident plane wave ψ_0 on the 2D surface \mathcal{N}_1 , with $n \times m$ cones: $n \times m$ conic singularities will diffract the incident wave in $n \times m$ -components. We want to evaluate the S-matrix from the in-state 0 to the out-the box one. One will expect that a fraction of initial probability density will escape from the box by the sides $\mathcal{N}_{1,2}, \mathcal{M}_{1,2}, \mathcal{P}_{1,2}$, another fraction will be trapped "forever" (for a time-life equal to the one of the system) inside the box. As a consequence, we have to consider all possible diffraction stories/paths. We also have to consider more complicated diffraction paths: the initial wave can scatter back and forth in the system before going-out.

We can consider the problem as a superposition of the initial wave function, assumed as a wave plane, and the diffracted wave functions for each conic singularities. In this system, we can label the position of all the conic singularities as (i, j, k) , where $i = 1, \dots, n$, $j = 1, \dots, m$, $k = 1, \dots, p$. The total wave function can be written as

$$\begin{aligned} \phi_0 + f(\mathbf{n}_0, \mathbf{n}_{111}) \frac{e^{ikr_{111}}}{r_{111}} + f(\mathbf{n}_0, \mathbf{n}_{121}) \frac{e^{ikr_{121}}}{r_{121}} + \dots + f(\mathbf{n}_0, \mathbf{n}_{1N1}) \frac{e^{ikr_{1N1}}}{r_{1N1}} \\ + f(\mathbf{n}_{111}, \mathbf{n}_{121}) \frac{e^{ikr_{121}}}{r_{121}} + \dots + f(\mathbf{n}_{111}, \mathbf{n}_{1N1}) \frac{e^{ikr_{11N}}}{r_{11N}} \\ + f(\mathbf{n}_{111}, \mathbf{n}_{211}) \frac{e^{ikr_{211}}}{r_{211}} + f(\mathbf{n}_{111}, \mathbf{n}_{221}) \frac{e^{ikr_{221}}}{r_{221}} + \dots + f(\mathbf{n}_{111}, \mathbf{n}_{2M1}) \frac{e^{ikr_{2M1}}}{r_{2M1}} + f(\mathbf{n}_{111}, \mathbf{n}_{212}) \frac{e^{ikr_{212}}}{r_{212}} \\ + \dots + f(\mathbf{n}_{111}, \mathbf{n}_{21P}) \frac{e^{ikr_{21P}}}{r_{21P}} + \dots + f(\mathbf{n}_{111}, \mathbf{n}_{2MP}) \frac{e^{ikr_{2MP}}}{r_{2MP}} + \dots \end{aligned} \quad (22)$$

where \mathbf{n}_0 is the wave vector of the incident plane wave, \mathbf{n}_{ijk} are wave vectors of the scattered waves from the conic singularities in positions ijk , r_{ijk} are radii from positions ijk .

Under this approximation, we can use the transition amplitudes of the one scattering problem considered in the previous section.

The resultant wave function will be a superposition of an infinite series of waves. As a consequence, the total wave function will be highly chaoticized by the superposition of all the scattered waves.

An S-matrix for one possible diffraction path is

$$\langle in | S^{1th-short} | out \rangle = S_{0-111} S_{111-222} S_{222-333} \dots S_{(n-1)(m-1)(p-1)-(nmp)} \quad (23)$$

where $S_{111-222}$ represents the S-matrix for a process from in-state (after a scattering on) 111 and with an out-state (after a scattering on) 222. This formulation can be consider if and only if the interdistances among singularities are much higher than the cones' sizes.

We can write a generic S-matrix for one diffraction path as

$$\langle in|S^{Kth}|out\rangle = S_{0-1jk}S_{ijk}S_{i'j'k'}\dots S_{(i^{n-1}j^{m-1}k^{p-1})-(i^nj^mk^p)} \quad (24)$$

(24) with conditions

$$i \leq i' \leq i + 1 \quad (25)$$

$$j \leq j' \leq j + 1 \quad (26)$$

$$k \leq k' \leq k + 1 \quad (27)$$

...

$$i^{n-1} \leq i^n \leq i^{n-1} + 1 \quad (28)$$

$$j^{m-1} \leq j^m \leq j^{m-1} + 1 \quad (29)$$

$$k^{p-1} \leq k^p \leq k^{p-1} + 1 \quad (30)$$

represent a class of paths similar to (23).

These class of paths are "minimal" ones: there are not back-transitions. "Minimal paths" are $n \times m \times p \times (n-1)$; while the number of non-minimal paths will diverge.

The total S-matrix is the (infinite) sum on all diffraction paths

$$\langle in|S_n^{OUT}|out\rangle = \sum_{paths} \langle in|S_n^{K-th}|out\rangle \quad (31)$$

The S-matrix for one diffraction path cn be written as

$$(S^{Kth})_n = \prod_{j=first}^{last} \frac{a_j J_{n\nu}(k_j \bar{r}_j) H_{n\nu}^-(k_j \bar{r}_j) + \frac{2i}{\pi}}{a_j J_{n\nu}(k_j \bar{r}_j) H_{n\nu}^+(k_j \bar{r}_j) - \frac{2i}{\pi}} \quad (32)$$

where the product is performed from the first scattering to the last one, and a_j, \bar{r}_j, k_j depend by the particular j-th conic singularity (k_j depends on the direction of the conic axis).

However let us remark that $S^{OUT} \neq S^{TOT}$: a part of the total S-matrix is associated to the trapped part of the wave function. Let us call this S-matrix S^{hidden} .

On the other hand, (32) takes only in consideration the continuos part of the S matrix, without resonant poles.

Bound states correspond to poles along negative real energies on the first Riemann sheet, of the resolvent operator $R(z) = (z - H)^{-1}$. In fact, the Hamiltonian is quadratic in momentum so that the inversed function $p = \sqrt{2mE}$ has a cut on the $[0, +\infty]$ axis, attaching two Riemann sheets. However, there will be also other poles at complex energies on the second Riemann sheet. The two poles correspond to $E_a = \mathcal{E}_a - i\Gamma_a/2$ and $E_a^* = \mathcal{E}_a + i\Gamma_a/2$, . i.e at the so called scattering resonances and anti-scattering resonances. In particular, $\mathcal{E}_a > 0$ is the real part of the energy while $\Gamma_a > 0$ corresponds to the resonances' widths. The dependence of the scattering amplitude on energy is strictly relates to these poles as

$$f(\mathbf{n}; E) \simeq f_c(\mathbf{n}; E) + \sum_r \frac{a_r(\mathbf{n})}{E - \mathcal{E}_r + i\Gamma_r/2} \quad (33)$$

where f_c is a smoothed amplitude corresponding to the continuous part of the spectrum while $a_r(\bar{n})$ are the residues of the resonances' poles. f_c corresponds to the S-matrix (31) in our case.

As a consequence, the resonants' parts of the amplitude will interfere among each other and with the non-resonant parts.

We can also generalize the notion of time delay also to the non-relativistic quantum chaotic mechanics

$$\mathcal{T}(E) = \frac{1}{i} \text{tr} \frac{d}{dE} \ln S(E) \quad (34)$$

Let us remind that in a theory with $H = H_0 + V$,

$$S(E) = 1 - 2\pi\delta(E - H_0)T(E + i0^+) \quad (35)$$

where T is the transition operator

$$T(z) = V + V \frac{1}{z - H} V \quad (36)$$

so that (37) can also be re-expressed in terms of the Hamiltonian as

$$\mathcal{T}(E) = -2\text{Im} \text{tr} \left(\frac{1}{E - H + V + i0^+} - \frac{1}{E - H_0 + i0^+} \right) = 2\pi\Delta D(E) \quad (37)$$

where $\Delta D(E)$ is the difference between the level densities of the total Hamiltonian and the asymptotic free one. This relation shows how the time delay is related by the resonance spectrum. Again, $\mathcal{T}(E)$ will diverge for bounds' states, so that this is an alternative way to define the bounds' spectrum. In appendix C, a discussion of chaotic spectrum of resonances in semiclassical limit are reviewed.

3.3. Comments on the range of validity of the previous calculations

The limitations of our approximated calculations shown in the previous section are understood and we resume the main relevant ones:

i) these calculations were done under the first order Born approximation. This approximation can be accepted if the interdistances among the conic geometries are much higher than the size of the cones. For interdistances comparable to cones' sizes, higher orders' corrections have to be considered.

ii) These calculations are based on simple non-relativistic quantum mechanics. In relativistic regime, obviously relativistic quantum field theory is the right framework to use.

Let us note that:

a) the disposition of conic singularities was assumed completely random. Otherwise, a chaotization of the quantum wave function is not generically expected: for a regular disposition of equally oriented conic singularities, one will expect a coherent superposition as in regular lattice, having in mind the Bragg's diffraction for example.

b) the problem becomes a trivial one if the wave length of the in-coming wave-function is comparable to the size of the system. In (32), this limit corresponds to $k_j \bar{r}_j \simeq 0$. So, we are assuming that λ is comparable to the size of conic geometries, and that conic geometries have sizes comparable each others.

3.3.1. Quantum field theories

In this section we will formally discuss the problem of the "box of cones" from a QFT point of view.

Let us return to our "box of cones" gedanken experiment. In this case, a formulation of the problem is again simpler than a realistic case: supposing interdistances much higher than cones' dimensions, In this case, we can define a transition amplitude for each cone. Let us suppose to be interested to calculate the transition amplitude for a field configuration ϕ_0 to a field configuration ϕ_N . ϕ_0 is the initial field configuration defined on a t_0 , before entering in the system, while ϕ_N is a field configuration of a time t_N , corresponding to a an out-going state from the system.

One example of propagation Path $0 - 111 - 222 - 333 - \dots - nmp - N$

$$\langle \phi_0, t_0 | \phi_{111, in}, t_{111, in} \rangle \langle \phi_{111, in}, t_{111, in} | \phi_{111, out}, t_{111, out} \rangle \langle \phi_{111, out}, t_{111, out} | \phi_{222, in}, t_{222, in} \rangle \quad (38)$$

$$\times \langle \phi_{222, in}, t_{222, in} | \phi_{222, out}, t_{222, out} \rangle \dots \langle \phi_{(n-1, m-1, p-1)}, t_{(n-1), (m-1), (p-1)} | \phi_{nmp}, t_{nmp} \rangle \langle \phi_{n, m, p}, t_{n, m, p} | \phi_N, t_N \rangle$$

where $|\phi_{ijk,in}, t_{ijk,in}\rangle$ and $|\phi_{ijk,out}, t_{ijk,out}\rangle$ are states before and after entering in the conic geometry ijk . In order to evaluate $\langle\phi_0, t_0|\phi_{nmp}, t_{nmp}\rangle$ one has to consider all the possible propagation paths from the initial position to the nmp -th conic singularity. Orders and summations are the analogous discussed for S-matrices in section 3.2. We define these amplitudes as

$$\langle\phi_{ijk}, t_{ijk}|\phi_{i'j'k'}, t_{i'j'k'}, in\rangle = \int_{\mathcal{M}_0} \mathcal{D}\phi e^{iI[\phi]} \quad (39)$$

while

$$\langle\phi_{ijk,in}, t_{ijk,in}|\phi_{ijk,out}, t_{ijk,out}\rangle = \int_{\mathcal{M}_{ijk}} \mathcal{D}\phi e^{iI[\phi]} \quad (40)$$

where \mathcal{M}_0 is the Minkowski space-time, while \mathcal{M}_{ijk} is the ijk -cone space-time. Again one can easily get that for a large system of naked conic singularities, it will exist a class of propagators' paths, reaching the out state $|\phi_N, t_N\rangle$ only for a time $t_N \rightarrow \infty$. A simple example can be the propagator paths

$$|\langle\phi_{ijk}, t_{ijk}|\phi_{i'j'k'}, t_{i'j'k'}\rangle|^2 |\langle\phi_{ijk}, t_{ijk}^{(1)}|\phi_{i'j'k'}, t_{i'j'k'}^{(1)}\rangle|^2 \dots |\langle\phi_{ijk}, t_{ijk}^{(\infty)}|\phi_{i'j'k'}, t_{i'j'k'}^{(\infty)}\rangle|^2 \quad (41)$$

where $t_{ijk}^\infty > \dots > t_{ijk}^{(1)} > t_{ijk}$ and $t_{i'j'k'}^\infty > \dots > t_{i'j'k'}^{(1)} > t_{i'j'k'}$. This amplitude is non-vanishing in such a system as an infinite sample of other ones. We can formally group these propagators in a $\langle BOX|BOX\rangle$ propagator, evaluating the probability that a field will remain in the box of cones after a time larger than the system life-time. On the other hand, let call $\langle BOX|OUT\rangle$ and $\langle OUT|OUT\rangle$ the other processes.

However, considering Standard Model fields (or its extensions), interactions among fields have to be considered inside the system. We can define an expectation value of a generic operator as

$$\langle\mathcal{O}\rangle = \sum_{\{\text{all K-paths}\}} \prod_{\{ijk, \text{path}\}} \langle ijk|\mathcal{O}^{Kth}|i'j'k'\rangle \quad (42)$$

where for example in a $0 - 111 - 222 - 333 - \dots - N$ path

$$\prod_{\{ijk, \text{path}\}} \langle ijk|\mathcal{O}^{Kth}|i'j'k'\rangle = \langle\phi_0, t_0|\mathcal{O}^{Kth}|\phi_{111,in}, t_{111,in}\rangle \langle\phi_{111,in}, t_{111,in}|\mathcal{O}^{Kth}|\phi_{111,out}, t_{111,out}\rangle \dots \quad (43)$$

and expectation values are evaluated on path integral on conic fixed backgrounds and non trivial geometries connecting cones. The formal way to introduce interactions' terms is $\mathcal{O} = \mathcal{L}_{int}$. For example, one can evaluate the expectation value of a $\frac{1}{4}\lambda\phi^4$ interaction term following the procedure (42). However, new interaction terms that usually have a zero expectation value in SM on a Minkowski space-time can be non-null in our system. For example, a photon scattering on a non-trivial background (especially in asperities among cones' connections) can decay into massive particles like for example inelastic processes as

$$\gamma + \langle G\dots G\rangle \rightarrow q\bar{q} + \langle G\dots G\rangle \rightarrow \text{hadrons} + \langle G\dots G\rangle$$

related to

$$\langle A_\mu \bar{q} \gamma^\mu q \rangle_{\text{Background}} \neq 0 \quad (44)$$

usually avoided by energy-momentum conservation in Minkowski space-time.

In non-relativistic limit, one can consider a non-relativistic path integral formulation. In bracket-notation, the propagator from (x_0, t_0) to (x_1, t_1) is

$$\mathcal{K}(x_0, t_0; x, t_1) = \langle x_0, t_0|x_1, t_1\rangle$$

This will be equivalent to wave functions' formulation considered in section 3.1. In this case, $\langle OUT|OUT\rangle$ will include all possible paths leading to the in-coming "ket" $|x_0, t_0\rangle$ to another "ket" out of the box. This a problem is chaotized: one has to consider the quantum interference of all paths for all conic geometries. An example among these paths is $0 - 111 - 222 - 333 - \dots - nmp - N$

$$\begin{aligned} & \langle x_0, t_0|x_{111,in}, t_{111,in}\rangle \langle x_{111,in}, t_{111,in}|x_{111,out}, t_{111,out}\rangle \langle x_{111,out}, t_{111,out}|x_{222,in}, t_{222,in}\rangle \\ & \times \langle x_{222,in}, t_{222,in}|x_{222,out}, t_{222,out}\rangle \dots \langle x_{(n-1,m-1,p-1)}, t_{(n-1),(m-1),(p-1)}|x_{nmp}, t_{nmp}\rangle \end{aligned} \quad (45)$$

where $|x_{ijk,in}, t_{ijk,in}\rangle$ and $|x_{ijk,out}, t_{ijk,out}\rangle$ are states incoming and outgoing "kets" in the conic geometry ijk .

An example of trapped propagators is

$$|\langle x_{ijk}, t_{ijk} | x_{i'j'k'}, t_{i'j'k'} \rangle|^2 |\langle x_{ijk}, t_{ijk}^{(1)} | x_{i'j'k'}, t_{i'j'k'}^{(1)} \rangle|^2 \dots |\langle x_{ijk}, t_{ijk}^{(\infty)} | x_{i'j'k'}, t_{i'j'k'}^{(\infty)} \rangle|^2 \quad (46)$$

where $t_{ijk}^\infty > \dots > t_{ijk}^{(1)} > t_{ijk}$ and $t_{i'j'k'}^\infty > \dots > t_{i'j'k'}^{(1)} > t_{i'j'k'}$. An ensemble of diffraction paths from OUT to BOX states will be chaotically attracted into trapped chaotic zones.

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- [1] J. D. Bekenstein, Phys. Rev. **D7** (1973) 2333.
 - [2] S. W. Hawking, Phys. Rev. D **13**, 2 (1976).
 - [3] L. Susskind, L. Thorlacius and J. Uglum, Phys. Rev. D **48** (1993) 3743 [hep-th/9306069].
 - [4] G. 't Hooft, Nucl. Phys. B **256** (1985) 727.
 - [5] S. L. Braunstein, S. Pirandola and K. Życzkowski, Phys. Rev. Lett. **110** (2013) 10, 101301 [arXiv:0907.1190 [quant-ph]].
 - [6] A. Almheiri, D. Marolf, J. Polchinski and J. Sully, JHEP **1302** (2013) 062 [arXiv:1207.3123 [hep-th]].
 - [7] P. Kraus and S. D. Mathur, arXiv:1505.05078 [hep-th].
 - [8] R. J. Gleiser and G. Dotti, Class. Quant. Grav. **23**, 5063 (2006) [arXiv:gr-qc/0604021]; G. Dotti, R. Gleiser and J. Pullin, Phys. Lett. B **644**, 289 (2007) [arXiv:gr-qc/0607052]; G. Dotti, R. J. Gleiser, I. F. Ranea-Sandoval and H. Vucetich, Class. Quant. Grav. **25** (2008) 245012 [arXiv:0805.4306 [gr-qc]]; G. Dotti and R. J. Gleiser, Class. Quant. Grav. **26** (2009) 215002 [arXiv:0809.3615 [gr-qc]].
 - [9] D. M. Eardley and L. Smarr, Phys. Rev. D **19** (1979) 2239; D. M. Eardley, NATO Sci. Ser. B **156** 229. P. S. Joshi and D. Malafarina, Phys Rev D **83**, 024009 (2011); P. S. Joshi and D. Malafarina, Gen. Rel. Grav. **45** (2), 305 (2013); S. Satin, D. Malafarina and P. S. Joshi, arXiv:1409.0505 [gr-qc]; L. Kong, D. Malafarina and C. Bambi, Eur. Phys. J. C **74** (2014) 2983 [arXiv:1310.8376 [gr-qc]]; P. S. Joshi, D. Malafarina and R. Narayan, Class. Quant. Grav. **31** (2014) 015002 [arXiv:1304.7331 [gr-qc]]; N. Ortiz, AIP Conf. Proc. **1473** (2012) 49 [arXiv:1204.4481 [gr-qc]]; P. S. Joshi and D. Malafarina, Int. J. Mod. Phys. D **20** (2011) 2641 [arXiv:1201.3660 [gr-qc]]; U. Miyamoto, S. Jhingan and T. Harada, arXiv:1108.0248 [gr-qc].
 - [10] A. Addazi, arXiv:1508.04054 [gr-qc].
 - [11] A. Addazi, arXiv:1510.05876 [gr-qc].
 - [12] B. S. DeWitt, Phys. Rev. **160** (1967) 1113.
 - [13] S. Carlip and C. Teitelboim, Class. Quant. Grav. **12** (1995) 1699 [gr-qc/9312002].
 - [14] R. Brustein and M. Hadad, Phys. Lett. B **718** (2012) 653 [arXiv:1202.5273 [hep-th]].
 - [15] R. M. Wald, Phys. Rev. D **48** (1993) 3427 [gr-qc/9307038].
 - [16] S. H. Shenker and D. Stanford, JHEP **1403** (2014) 067 [1306.0622]; L. A. Pando Zayas and C. A. Terrero-Escalante, JHEP **1009** (2010) 094 [arXiv:1007.0277 [hep-th]].
 - [17] F. Haake, *Quantum Signatures of Chaos* Edition: 2, Springer, 2001, ISBN 3-540-67723-2, ISBN 978-3-540-67723-9; M. Berry, *Quantum Chaology*, pp 104-5 of *Quantum: a guide for the perplexed* by J-Al Khalili.
 - [18] T. Tél, M. Gruinz, *Chaotic Dynamics*, Cambridge University Press (2006), Cambridge UK; J.M. Seoane and M. A. F. Sanjuán 2013 Rep. Prog. Phys. **76** 016001;
 - [19] V.I. Arnold, Russ.Math.Surv. **18**:6:85-191; J.K.Moser, Mem.Am.Math.Soc. **81**: 1-60.
 - [20] B.Eckhardt, J.Phys.A: Math. Gen. **20**: 5971-5979. P.Gaspard and S.A. Rice, J.Chem.Phys. **90**:2225-2241; P.Gaspard, *Chaos, Scattering and Statistical Mechanics*, Cambridge University Press, Cambridge UK.
 - [21] K.T. Hansen, Nonlinearity **6**: 753-770; M.M. Sano, J.Phys. A: Math. Gen. **27**: 4791-4803.
 - [22] A. Addazi and S. Capozziello, Int. J. Theor. Phys. **54** (2015) 6, 1818 [arXiv:1407.4840 [gr-qc]].
 - [23] A. Addazi and G. Esposito, Int. J. Mod. Phys. A **30** (2015) 1550103 [arXiv:1502.01471 [hep-th]].
 - [24] A. Addazi, arXiv:1505.07357 [hep-th].
 - [25] A. Addazi and M. Bianchi, JHEP **1412** (2014) 089 [arXiv:1407.2897 [hep-ph]].
 - [26] A. Addazi, JHEP **1504** (2015) 153 [arXiv:1501.04660 [hep-ph]].
 - [27] A. Addazi and M. Bianchi, arXiv:1502.01531 [hep-ph].
 - [28] A. Addazi and M. Bianchi, JHEP **1506** (2015) 012 [arXiv:1502.08041 [hep-ph]].
 - [29] A. Addazi, arXiv:1504.06799 [hep-ph].
 - [30] A. Addazi, arXiv:1505.00625 [hep-ph].
 - [31] A. Addazi, arXiv:1505.02080 [hep-ph].

- [32] A. Addazi, arXiv:1506.06351 [hep-ph].
- [33] A. Addazi, M. Bianchi and G. Ricciardi, arXiv:1510.00243 [hep-ph].
- [34] A. Addazi, arXiv:1510.02911 [hep-ph].
- [35] D.V. Fursaev, S. N. Solodukhin. arXiv preprint hep-th/9501127 (1995); D.V. Fursaev Nuclear Physics B **524.1** (1998): 447-468; D.V. Fursaev, Dmitri V. Physics Letters B **334.1** (1994): 53-60; D.V. Fursaev, G. Miele. "Cones, spins and heat kernels." Nuclear Physics B **484.3** (1997): 697-723.
- [36] E. Battista, E. Di Grezia and G. Esposito, Int. J. Geom. Meth. Mod. Phys. **12** (2015) 1550060 [arXiv:1410.3971 [gr-qc]].
- [37] K. S. Virbhadra, D. Narasimha and S. M. Chitre, Astron. Astrophys. **337** (1998) 1 [astro-ph/9801174].
- [38] K. S. Virbhadra and G. F. R. Ellis, Phys. Rev. D **62** (2000) 084003 [astro-ph/9904193].
- [39] K. S. Virbhadra and C. R. Keeton, Phys. Rev. D **77** (2008) 124014 [arXiv:0710.2333 [gr-qc]].
- [40] H. B. Nielsen and P. Olesen, Nucl.Phys. B**61**, 45 (1973);
A. Vilenkin, Phys.Rev. D**23**, 852 (1981).
- [41] P. Laguna and D. Garfinkle, Phys.Rev. D**40**, 1011 (1989);
M. Christensen, A. L. Larsen, and Y. Verbin, Phys.Rev. D**60**, 125012 (1999), arXiv:gr-qc/9904049 [gr-qc];
M. E. Ortiz, Phys.Rev. D**43**, 2521 (1991).

Appendix A: generalized Wheeler-De Witt equation

In this section, we report formal details and definitions of Carlip-Teitelboim approach for BH [13], based on an extension of the Wheeler-DeWitt equation [12]. This approach starts from first axioms of quantum mechanics applied to Wald formalism [15]. The idea was also well developed in [14].

This approach starts from a BH spacetime foliation with constant hypersurfaces Σ , with a space-like normal vector n_a . Conveniently, one can define a metric on the hypersurface Σ as

$$g_{ab} = h_{\gamma\delta} e_a^\gamma e_b^\delta$$

where e_a^γ are the basis vector of the tangent bundle; γ, δ -indices are the so-called induced coordinates. Let us consider a section of the hypersurface Σ with surface

$$A = -\frac{1}{2} \int_{\Sigma} dS$$

with

$$dA = a\epsilon$$

and a the area element and $\epsilon = \nabla n$ (we omit indices of ϵ, n, ∇).

We can conveniently use re-definitions of A and its Lie derivative $\mathcal{L}_n A$ (on the direction n), in term of hypersurface metric and normal vector:

$$A = -\frac{1}{2} \int_{\Sigma} h^{\gamma\delta} n^a n^b \epsilon_{\gamma a} \epsilon_{\delta b}$$

$$\mathcal{L}_n A = \int_{\Sigma} (\mathcal{L}_n h^{\gamma\delta}) n^a n^b a \epsilon_{\gamma a} \epsilon_{\delta b}$$

From these one could derive the following final equation [14]:

$$\left\{ -\frac{1}{2} \frac{\mathcal{L}_n A}{A}(t_0), \frac{1}{2\pi} S_w(t_1) \right\} = \frac{1}{\sqrt{-g_{00}}} \delta(t_0 - t_1)$$

where S_w is the Wald Noether charge entropy, formally defined as

$$S_w = -2\pi \int_H \frac{\partial L}{\partial R_{\gamma a \delta b}} a \epsilon_{\gamma a} \epsilon_{\delta b}$$

integrated on the Horizon surface. (L has not to be confused with Lie derivative, because it is just the lagrangian density).

For a stationary space-time metric, the (D-1)-dimensional hypersurface is the product of the proper time τ and the (D-2)-dimensional hypersurface $A_{D-1} = \tau A_{D-2}$. But $\mathcal{L}_n A_{D-1} = 0$, implying

$$A_{D-2} \mathcal{L}_n \tau + \tau \mathcal{L}_n A_{D-2} = 0$$

This allows us to express the Lie derivative of the Area $\mathcal{L}_n A_{D-2}$ in term of the one of the proper time $\mathcal{L}_n \tau$. However, the so-called opening angle at the horizon is

$$\Theta = \frac{1}{2} \mathcal{L}_n \tau$$

so that we can relate Θ to the Lie derivative of the area as

$$\Theta = -\frac{\tau}{2} \frac{\mathcal{L}_n A_{D-2}}{A_{D-2}}$$

From these relations, we can arrive to a very suggestive one [14]

$$\left\{ \Theta, \frac{1}{2\pi} S_w \right\} = 1$$

where $\{\dots\}$ is the Poisson bracket. As usually done from classical mechanics to quantum mechanics, one could quantize a Black hole as $\{\dots\} \rightarrow \frac{i}{\hbar} [\dots]$. This leads to a Schrodinger equation for the wave function Ψ as

$$\frac{\hbar}{i} \delta \Psi(X) + \delta X \Psi(X) = 0$$

where $X = (\Theta, T, \dots)$, and

$$\delta X = [\delta T M + \delta \Theta \frac{1}{2\pi} S_w + dC]$$

where $dC = p\delta q$ represents all the possible variation of conjugate variables associated to conserved Noether charges; T is the time separation at infinity, M is the ADM mass. The equation (2) describes Ψ only with respect to S_w and not other variables. This is a particular case of the one discussed here.

Appendix B: euclidean path integral reformulation

In this section, we will reformulate definition in section 2 in path integral language [10, 11].

Let us consider a system of N conic naked horizonless singularities, inside a box \mathcal{M} , with a surface $\partial\mathcal{M}$. This system satisfied the following hypothesis:

I) Partition functions Z_I for each metric tensor $g^{I=1,\dots,N}$ can be formally defined, with N metrics are considered in thermal equilibrium with the box.

II) The leading order of the total partition function Z_{TOT} is the product of each single partition functions, as

$$Z_{TOT} = \prod_{I=1}^N Z_I \quad (47)$$

This approximation can be reasonably trusted if intergeometries' interactions are negligible (with respect to the temperature scale of the box).

III) The total average partition function will be

$$\langle Z_{TOT} \rangle = e^{-\frac{\beta^2}{16\pi} - \frac{\sigma_\beta^2}{16\pi}} = Z_E e^{-\frac{\sigma_\beta^2}{16\pi}} \quad (48)$$

where Z_E is the semiclassical euclidean partition function of a semiclassical black hole, σ_β the variance of β -variable. This leads to an entropy

$$\langle S \rangle = \frac{\beta^2}{16\pi} + \frac{\sigma_\beta^2}{16\pi} \quad (49)$$

Appendix C: semiclassical chaotic scattering

In this section, we review some aspects of semiclassical chaotic scattering considered in our previous papers [10, 11]. The semiclassical propagator can be written as

$$\mathcal{K}_{WKB}(\mathbf{r}, \mathbf{r}_0, t) \simeq \sum_n \mathcal{A}_n(\mathbf{r}, \mathbf{r}_0, t) e^{\frac{i}{\hbar} I_n} \quad (50)$$

summed on all over the classical n-orbits inside our billiard; amplitudes \mathcal{A}_n are defined as

$$\mathcal{A}_n(\mathbf{r}, \mathbf{r}_0, t) = \frac{1}{(2\pi i \hbar)^{\nu/2}} \sqrt{|det[\partial \mathbf{r}_0 \partial \mathbf{r}_0 I_n[\mathbf{r}, \mathbf{r}_0, t]]|} e^{-\frac{i\pi h_n}{2}} \quad (51)$$

(h_n counts for the the number of conjugate points along the n-th orbit). The amplitude is related to Lyapunov exponents as

$$|\mathcal{A}_n| \sim |t|^{-\nu/2} \quad (52)$$

on stable orbits

$$|\mathcal{A}_n| \sim \exp\left(-\frac{1}{2} \sum_{\lambda_k > 0} \lambda_k t\right) \quad (53)$$

on unstable ones.

In our chaotic system, we expect many resonances. A spectrum of resonances called Pollicott-Ruelle ones characterizes the chaotic dynamics of our billiard. As a consequence, transitions or survival probabilities are averaged over the large number resonances.

So that, a wavepacket ψ_0 in a region R (ν -dimensional space) has a quantum survival probability

$$\mathcal{P}(t) = \text{tr} \mathcal{I}_V(\mathbf{r}) e^{-\frac{iHt}{\hbar}} \rho_0 e^{+\frac{iHt}{\hbar}} \quad (54)$$

where the initial density matrix $\rho_0 = |\psi_0\rangle\langle\psi_0|$, \mathcal{I}_V is zero for resonances \mathbf{r} out of the region V , and $\mathbf{1}$ for resonances into V . In semiclassical approximation, the (survival) probability is

$$\begin{aligned} \mathcal{P}(t) &\simeq \int \frac{d\Gamma_{ph}}{(2\pi\hbar)^f} \mathcal{I}_D e^{\mathbf{L}_{cl}t} \tilde{\rho}_0 + O(\hbar^{-\nu+1}) \\ &+ \frac{1}{\pi\hbar} \int dE \sum_e \sum_r \frac{\cos(r \frac{S_e}{\hbar} - r \frac{\pi}{2} \mathbf{m}_e)}{\sqrt{|det(\mathbf{m}_e^r - \mathbf{1})|}} \int_e \mathcal{I}_D \tilde{\rho}_0 \text{Exp}\{\mathbf{L}_{cl}t\} dt + O(\hbar^0) \end{aligned} \quad (55)$$

where $d\Gamma_{ph} = d\mathbf{p}d\mathbf{r}$ is the phase space infinitesimal volume and the sum is on all the periodic orbits as mentioned above (primary elementary periodic orbits are labelled in e while the number of their repetitions r); \mathbf{m}_e is the Maslov index, $S_e(E) = \int \mathbf{p} \cdot d\mathbf{r}$, $\tau_e = \int_E S_e(E)$, \mathcal{M} is the Poincaré map in the neighborhood of the r-orbit (it is a $(2\nu - 2) \times (2\nu - 2)$ matrix); \mathbf{L}_{cl} is the Liouvillian operator defined as $\mathbf{L}_{cl} = \{H_{cl}, \dots\}_{Poisson}$; $\tilde{\rho}_0$ is the Wigner transform of the (initial) density state. In particular, \mathbf{L}_{cl} defines the Pollicott-Ruelle peaks mentioned above:

$$\mathbf{L}_{cl} \phi_m = \{\mathbf{H}_{cl}, \phi_m\}_{Poisson} = \lambda_m \phi_m \quad (56)$$

where eigenvalues λ_m are complex ones and they correspond to P.R. spectrum, where eigenstates ϕ_m are an ortonormal basis composed of Gelfand-Schwartz distributions. $\text{Re}(\lambda_m) \leq 0$, correspond to an ensamble bounded periodic orbits; while $\text{Im}(\lambda_n)$ correspond to decays and instabilities in the system. Despite of the complicated form of (55) the survival probability has a leading order $\mathcal{P}(t) \sim e^{-\gamma(E)t}$, where $\gamma(E)$ is the classical escape probability (from the system). This leading order can be obtained by the 0th order of the expansion

$$\mathcal{P}(t) \simeq \int \sum_m \langle \mathcal{I}_V | \phi_m(E) \rangle \langle \tilde{\phi}_m(E) | e^{\lambda_m(E)t} | \phi_m(E) \rangle \langle \tilde{\phi}_m(E) | \tilde{\rho}_0 \rangle \quad (57)$$